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DISCRETE INTERACTION OF A PLATE WITH A SEMI-INFINITE STIFFENER

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The plane contact problem of interaction between an infinite plate and a semi-infinite stiffener through an infinite system of stiff circular inclusions (rivets) is considered. The problem reduces to an infinite system of linear algebraic equations with coefficients dependent on the difference in the indices; the exact solution of this system is constructed by reducing it to a known Riemann-Hilbert problem by the method of Fel'd [1]. This problem can be considered as a discrete analog of the problem of continuous interaction between a plate and a semi-infinite stiffener [2].

In computing the strength of riveted panels in which the interaction between thin plate and one-dimensional reinforcing elements is realized through a discrete system of rivets, it is usually assumed that the plate and the reinforcing elements interact along the whole line of contact (continuous interaction). Such an approach is legitimate for a part of the rivet arrangement, when taking account of the discrete nature of the interaction between the panel elements plays no essential role. A number of problems on the continuous interaction between infinite and semi-infinite plates and finite, semi-infinite, and infinite stiffeners is considered in [2—6], etc.

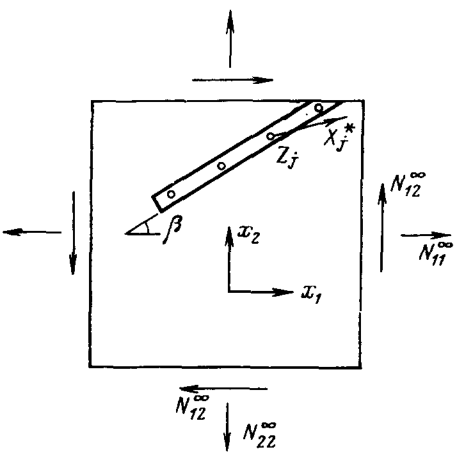

Fig. 1

A solution of the problem on the transmission of the force from an infinite stiffener to an infinite plate through a periodic system of rivets is given in [7]. An exact solution of the more difficult problem of the interaction between an infinite plate and a semi-infinite stiffener attached to the plate by using an infinite number of rivets arranged at an identical distance from each other is given below.

1. Formulation of the problem. Let a semi-infinite stiffener of cross section F_0 be riveted to an infinite thin plate at an angle β to the ox_1-axis (x_1, x_2 are rectangular Cartesian coordinates in the middle plane of the plate, $z = x_1 + ix_2$ is the complex variable). The rivets are arranged with the constant spacing R and have the identical radius r (Fig. 1).

Let us assume the following: (1) there is no friction between the plate and the stiffener. (2) We neglect the effect of eccentric attachment of the stiffener (relative to the middle plane of the plate). (3) A plane state of stress is realized in the plate, and the rivets in the plate are modelled by stiff circular inclusions. (4) The stiffener works only in tension-compression, where its attenuation because of installation of the rivets is not taken into account.

Let us limit ourselves to an examination of the case when the plate is subjected at infinity to constant forces characterized by the tensor $N_{\alpha\beta}^{\infty}$ $(\alpha, \beta = 1, 2)$, and point forces $X_j^* = X_{1j}^* + iX_{2j}^*$ $(j = 0, 1, 2, \ldots$ are applied to the centers of the rivets z_j.

Let us imagine the stiffener separated from the plate and let us apply unknown interaction forces $X_j = X_{1j} + iX_{2j}$ and $-X_j$, respectively, to the rivet centers z_j of the plate and the stiffener. Let N_j be the force in a rod at the section between the j-th and $(j+1)$-th rivets. It follows from the equilibrium of the part of the stiffener in the neighborhood of the j-th rivet

$$X_j = X_j^* + (N_j - \theta_{j-1}N_{j-1})e^{i\beta} \quad (j = 0, 1, 2, \ldots) \tag{1.1}$$

$$\theta_j = \begin{cases} 1 & \text{for } j = 0, 1, 2, \ldots \\ 0 & \text{for } j = -1, -2, \ldots \end{cases}$$

Taking account of the notation

$$P_j + iQ_j = X_j e^{-i\beta}, \quad P_j^* + iQ_j^* = X_j^* e^{-i\beta} \quad (j = 0, 1, 2, \ldots)$$

the relationships (1.1) can be given the form

$$P_j = P_j^* + N_j - \theta_{j-1}N_{j-1} \tag{1.2}$$

$$Q_j = Q_j^* \quad (j = 0, 1, 2, \ldots)$$

Hence, it follows in particular that

$$N_k = \sum_{j=0}^{k} (P_j - P_j^*) \quad (k = 0, 1, 2, \ldots) \tag{1.3}$$

In order to determine the unknown forces P_j transmitted through the rivets, we write the conditions for compatibility of the mutual displacements of the adjacent rivets in the plate and the stiffener. Let γ_k be the mutual displacement of the k-th and $(k+1)$-th rivets in the plate in the stiffener direction. Then

$$\gamma_k = \operatorname{Re}\{[w(z_{k+1}) - w(z_k)]e^{-i\beta}\} = \frac{RN_k}{E_0F_0} \quad (k = 0, 1, 2, \ldots) \tag{1.4}$$

$$(1 - \nu)Bw(z) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \tag{1.5}$$

$$(B = Eh / (1 - \nu^2),\ \varkappa = (3 - \nu) / (1 + \nu))$$

where the relationship (1.5) to determine the complex displacement vector $w = u_1 + iu_2$ is written in conformity with [8], h is the plate thickness, ν, E are the Poisson ratio and Young's modulus, respectively, of the plate material, $\varphi(z)$ and $\psi(z)$ are complex potentials, and E_0 is the Young's modulus of the stiffener material.

To find the functions $\varphi(z)$ and $\psi(z)$ we must solve the plane problem of elasticity theory for a plane with an infinite number of stiff circular inclusions simulating rivets. Hovever, in the majority of cases of practical importance, the rivet radius is small compared to their spacing $(r / R = 0.02—0.1)$. This permits us to limit ourselves to taking account of the asymptotic interaction between the inclusions and to use the superposition principle in the following form:

$$\varphi(z) = \varphi_*(z) + \sum_{j=0}^{\infty} \varphi_j(z), \quad \psi(z) = \psi_*(z) + \sum_{j=0}^{\infty} \psi_j(z) \tag{1.6}$$

Here

$$\varphi_*(z) = \Gamma z, \ \psi_*(z) = \Gamma' z \tag{1.7}$$

$$\varphi_j(z) = -\frac{X_j}{2\pi(1+\varkappa)} \ln(z - z_j)$$

$$\psi_j(z) = \frac{\varkappa \bar{X}_j}{2\pi(1+\varkappa)} \ln(z - z_j) + \frac{X_j}{2\pi(1+\varkappa)} \left[\frac{\bar{z}_j}{z - z_j} + \frac{r^2}{(z - z_j)^2}\right]$$

$$(j = 0, 1, 2, \ldots; \ 4\Gamma = N_{11}^{\infty} + N_{22}^{\infty}; \ 2\Gamma' = N_{22}^{\infty} - N_{11}^{\infty} + 2iN_{12}^{\infty})$$

where $\varphi_*(z)$, $\psi_*(z)$ are the potentials of the given homogeneous external field, $\varphi_j(z)$, $\psi_j(z)$ are the potentials for a plane with one j-th stiff circular inclusion to whose center a force X_j is applied.

Taking into account that $(r / R)^2 \ll 1$, after certain manipulations the compatibility condition (1.4) can be reduced by using the relationships (1.5)—(1.7) to the form

$$\gamma_0 - \sum_{j=0}^{\infty} \Gamma_{k-j} P_j = \frac{N_k}{\omega} \quad (k = 0, 1, 2, \ldots) \tag{1.8}$$

Here

$$\gamma_0 = \frac{4\pi R}{1+\nu}\left[\frac{1-\nu}{1+\nu}(N_{11}^{\infty} + N_{22}^{\infty}) + (N_{11}^{\infty} - N_{22}^{\infty})\cos 2\beta + 2N_{12}^{\infty} \sin 2\beta\right] \tag{1.9}$$

$$\Gamma_k = -\Gamma_{-k-1} = \begin{cases} 1 + 2\varkappa \ln \varepsilon, & k = -1 \\ -1 - 2\varkappa \ln \varepsilon, & k = 0 \\ 2\varkappa \ln(1 + 1/k), & k \neq 0, -1 \end{cases} \tag{1.10}$$

$$\omega = \frac{(1+\nu) E_0 F_0}{8\pi(1-\nu) BR}, \quad \varepsilon = r/R$$

Subtracting the ($k - 1$)-th equation from the k-th in the system (1.8) (for $k = 1$, $2, \ldots$ and taking (1.2) into account, we find

$$P_0 + \omega \sum_{j=0}^{\infty} \Gamma_{-j} P_j = \omega\gamma_0 + P_0^* \tag{1.11}$$

$$P_k + \omega \sum_{j=0}^{\infty} b_{k-j} P_j = P_k^* \quad (k = 1, 2, \ldots)$$

where

$$b_k = b_{-k} = \Gamma_k + \Gamma_{-k} = \begin{cases} 2\Gamma_0, & k = 0 \\ 1 + 2\varkappa \ln 2\varepsilon, & |k| = 1 \\ 2\varkappa \ln(1 - k^{-2}), & |k| = 2, 3, \ldots \end{cases} \tag{1.12}$$

Equations (1.11) are used to determine the forces P_j transmitted through the rivets. The forces in the stiffener N_j are then found from (1.3). The field in the plate is determined easily in terms of the P_j found by means of the formulas (1.1), (1.2), (1.6) and (1.7).

2. Solution of the infinite algebraic system. Following the method proposed by Fel'd [1], let us consider the functions

$$B(z) = \sum_{k=-\infty}^{\infty} b_k z^k, \quad \Gamma(z) = \sum_{k=-\infty}^{\infty} \Gamma_k z^k \tag{2.1}$$

Since

$$\lim_{k\to\infty} |b_k|^{1/k} = \lim_{k\to\infty} [2\varkappa\,|\ln(1-k^{-2})|]^{1/k} = 1$$

and the series

$$b(\tau) = B(e^{i\tau}) = 2\sum_{k=1}^{\infty} b_k(\cos k\tau - 1) \tag{2.2}$$

converges, then the function $B(z)$ is regular in the unit circle C. Moreover, by using (1.10) and (1.12) it can be shown that

$$B(z) = (1-z)\Gamma(z)$$

Let us assume that the functions

$$P^+(z) = \sum_{k=0}^{\infty} P_k z^k, \quad P^*(z) = \sum_{k=0}^{\infty} P_k{}^* z^k$$

are analytic in the domain $D_+ + C$ (D_+ and D_- are the domains inside and outside the unit circle, respectively). This assumption is justified in the ratio of the functions $P^+(z)$ after the problem has been solved. The analyticity of the function $P^*(z)$ in the domain $D_+ + C$ is assured by the nature of the external force distribution $P_k{}^*$ under consideration.

It follows from this discussion that

$$P_k = \frac{1}{2\pi i}\int_C P^+(\zeta)\zeta^{-k-1}d\zeta = P^{+(k)}(0)/(k!) \quad (k=0,1,2,\ldots) \tag{2.3}$$

where

$$\frac{1}{2\pi i}\int_C P^+(\zeta)\zeta^{-k-1}d\zeta = \begin{cases} P_k, & k=0,1,2,\ldots \\ 0, & k=-1,-2,\ldots \end{cases} \tag{2.4}$$

Analogous relationships hold for the quantities $P_k{}^*$ and $P^*(z)$.

Taking account of these expressions, (1.11) becomes

$$\frac{1}{2\pi i}\int_C \frac{1+\omega\Gamma(\zeta)}{\zeta} P^+(\zeta)\,d\zeta = \omega\gamma_0 + P_0{}^* \tag{2.5}$$

$$\frac{1}{2\pi i}\int_C \frac{G(\zeta)P^+(\zeta) - P^*(\zeta)}{\zeta^{k+1}}\,d\zeta = 0 \quad (k=1,2,\ldots) \tag{2.6}$$

Here

$$G(\zeta) = 1 + \omega B(\zeta) = 1 + \omega(1-\zeta)\Gamma(\zeta) \tag{2.7}$$

Let us introduce the piecewise-analytic function

$$P(z) = \begin{cases} P^+(z), & z \in D_+ \\ P^-(z), & z \in D_- \end{cases}$$

which is a solution of the Riemann-Hilbert boundary value problem

$$P^-(\zeta) = G(\zeta)P^+(\zeta) - P^*(\zeta) \quad (\zeta \in C) \tag{2.8}$$

Then conditions (2.4) and (2.6) will be satisfied because of the definition of the function $P(z)$ and taking account of (2.7) and (2.8), Eq. (2.5) goes over into the relationship

$$P^+(1) = \omega\gamma_0 + P^*(1) \tag{2.9}$$

which reflects the equilibrium condition for the semi-infinite stiffener separated from the plate.

According to (2.2) and (2.7), the function $G(\zeta)$ $(\zeta \in C)$ is real and positive so that its index in the circle C is zero. Hence, the solution of the problem (2.8) has the form [9]

$$P^\pm(z) = X^\pm(z)[f^\pm(z) + P^-(\infty)], \quad X^\pm(z) = \exp[-F^\pm(z)] \tag{2.10}$$

$$F^\pm(z) = \frac{1}{2\pi i}\int_C \frac{\ln G(\zeta)}{\zeta - z}d\zeta, \quad f^\pm(z) = \frac{1}{2\pi i}\int_C \frac{P^*(\zeta)\,d\zeta}{X^-(\zeta)(\zeta - z)}$$

Finding the constant $P^-(\infty)$ from the equilibrium equation (2.9)

$$P^-(\infty) = \frac{\omega\gamma_0 + P^*(1)}{X^+(1)} - f^+(1)$$

we finally write the solution as follows:

$$P^\pm(z) = \left[\frac{\omega\gamma_0 + P^*(1)}{X^+(1)} - f^+(1) + f^\pm(z)\right]X^\pm(z)$$

According to (1.7) we hence find

$$P_k = \left[\frac{\omega\gamma_0 + P^*(1)}{X^+(1)} - f^+(1)\right]\frac{X^{+(k)}(0)}{k!} + \frac{1}{k!}[f^+(z)\,X^+(z)]_{z=0}^{(k)} \tag{2.11}$$

$(k = 0, 1, 2, \ldots)$

In conclusion, let us note a more convenient form of writing the last equations for calculations. To do this, we use the following recursion relations:

$$\frac{X^{+(k)}(z)}{k!} = -\frac{1}{k}\sum_{m=0}^{k-1}(k-m)\frac{X^{+(m)}(z)\,F^{+(k-m)}(z)}{m!\,(k-m)!} \quad (k = 1, 2, \ldots) \tag{2.12}$$

$$\frac{[f^+(z)\,X^+(z)]^{(k)}}{k!} = \sum_{m=0}^{k}\frac{f^{+(m)}(z)\,X^{+(k-m)}(z)}{m!\,(k-m)!} \quad (k = 0, 1, 2, \ldots) \tag{2.13}$$

and the notation

$$p_k = \frac{1}{k!}X^{+(k)}(0), \quad \delta_k = \frac{1}{k!}[f^+(z)\,X^+(z)]_{z=0}^{(k)} \tag{2.14}$$

$$\lambda_k = \frac{F^{+(k)}(0)}{k!} = \frac{1}{2\pi i}\int_C \frac{\ln G(\zeta)}{\zeta^{k+1}}d\zeta = \tag{2.15}$$

$$\frac{1}{\pi}\int_0^\pi \ln g(\sigma)\cdot\cos k\sigma d\sigma \quad (g(\sigma) = G(e^{i\sigma}))$$

$$\eta_k = \frac{f^{+(k)}(0)}{k!} = \frac{1}{2\pi i}\int_C \frac{P^*(\zeta)\,d\zeta}{X^-(\zeta)\,\zeta^{k+1}} \tag{2.16}$$

Taking into account that

$$p_0 = X^+(0) = \exp[-F^+(0)], \quad X^+(1) = \exp[-F(1)] \tag{2.17}$$

$$F^+(0) = \lambda_0, \quad F(1) = \frac{1}{2\pi i}\int_C \frac{\ln G(\zeta)}{\zeta - 1}\, d\zeta = \frac{\lambda_0}{2}$$

we obtain by using (2.11)–(2.16)

$$P_k = \left[\frac{\omega\gamma_0 + P^*(1)}{\sqrt{p_0}} - f^+(1)\right] p_k + \delta_k \quad (k = 0, 1, 2, \ldots) \tag{2.18}$$

where

$$p_k = -\frac{1}{k}\sum_{n=1}^{k} n p_{k-n}\lambda_n \quad (k = 1, 2, \ldots) \tag{2.19}$$

$$\delta_k = \sum_{m=0}^{k} p_{k-m}\eta_m \quad (k = 0, 1, 2, \ldots) \tag{2.20}$$

3. Numerical results. As an illustration of the general solution of the particular case when there is no external field, i.e. $N_{\alpha\beta}^{\infty} = 0$ and $\gamma_0 = 0$, but the force $X_0^* = e^{i\beta}$ is applied to the outermost rivet $(k = 0)$. Then $P_0^* = P^*(z) = 1$, $f^+(z) = 0$, $\eta_k = 0$ and according to (2.18) and (2.20)

$$P_k = p_k / \sqrt{p_0} \quad (k = 0, 1, 2, \ldots$$

It is now required to find the quadrature (2.15) and to use the recursion formula (2.19).

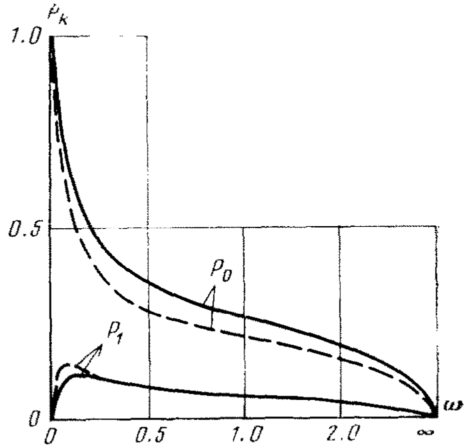

Fig. 2

Curves of the dependence of the forces P_0 and P_1 at the two outermost rivets on the relative stiffness parameter ω are represented for two values of the parameter ε equal to 0.02 (dashes) and 0.1 (solid line) in Fig. 2 for $\nu = 1/3$. A domain of values of ε encountered in practice lies between these values. The calculations were performed on the M-222M computer; the computation time was around two minutes (for 17 points on each curve).

Let us note that a variable scale is used along the ω axis in Fig. 2: to the left of 1 are values of the parameter ω itself, while to the right of 1 are values of $1/\omega$ so that for $\omega \leqslant 1$ the points ω and $1/\omega$ are symmetric relative to the point $\omega = 1$.

The dependence of P_k for $k = 2, 3, \ldots$ on the parameter ω qualitatively duplicates the curve for P_1 in Fig. 2. An appraisal of the distribution of the forces P_k over the rivets can be made from the following results (we present the values $10^4 \cdot P_k$, $k = 0, 1, \ldots, 5$):

ε	ω	$k=0$	1	2	3	4	5
	0.1	6325	1162	526	294	185	126
0.1	1.0	2617	566	376	280	222	182
	10	871	124	91	74	63	55
	0.1	5360	1397	651	355	216	142
0.02	1.0	2089	537	369	277	219	179
	10	689	112	84	69	59	51

Let us note that in the presence of one external field when $N_{\alpha\beta}^{\infty} \neq 0$ (and $X_k^* = 0$), we have

$$P_k / \gamma_0 = \omega p_k / \sqrt{p_0} \qquad (k = 0, 1, 2, \ldots)$$

which agrees with the case just considered to the accuracy of a factor of ω.

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